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Real simple n-Lie algebras admitting metric structures*

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Abstract

We give the multiplication structures of all real simple n-Lie algebras and prove that each of them has metric dimension 1 or 2 depending on that it belongs to type I or type II. We also determine the signatures of metrics on all real simple n-Lie algebras. Moreover, we present an example of real 3-Lie algebras which is indecomposable but has much larger metric dimension.

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1. Introduction

Recently, it has been shown that the structure of metric the 3-Lie algebras is closely linked to the supersymmetry and gauge symmetry transformations of the worldvolume theory of multiple coincident M2-branes (see [1–3]). The Bagger–Lambert theory has a novel local gauge symmetry which is not based on a Lie algebra, but rather on a 3-Lie algebra. It was found that the Jacobi identity for a 3-Lie algebra is essential to define the action with $N = 8$ supersymmetry, and the Jacobi identity can also be thought of as a generalized Plucker relation in the physics literature. To obtain the correct equations of motion for the Bagger–Lambert theory from a Lagrangian that is invariant under all aforementioned symmetries seems to require the 3-Lie algebra to admit an invariant inner product (i.e. a metric). The signature of this metric determines the relative signs of the kinetic terms for scalar and fermion fields in the Bagger–Lambert Lagrangian (see [1–3]). In ordinary gauge theory, a positive-definite metric is required in order to ensure that the theory has positive-definite kinetic terms and to prevent violations of unitarity due to propagating ghost-like degrees of freedom. However, there are very few 3-Lie algebras which admit positive-definite metrics. In fact, it has been shown (see

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[4, 5]) that all finite-dimensional 3-Lie algebras with positive-definite metrics are the direct sums of A_4^0 (see section 3) with trivial algebras. On the other hand, in order to find new interacting Bagger–Lambert Lagrangians and despite the possibility of negative-norm states, one is led to contemplate 3-Lie algebras with metrics having any signatures (p, q), such as $p = 1$ (Lorentzian), $p = 2$ or with a degenerate invariant symmetric bilinear forms, since in certain dynamical systems a zero-norm generator corresponds to a gauge symmetry while a negative-norm generator corresponds to a ghost (see [6–10]). Thus, it seems to be worthwhile and interesting, in both physical and mathematical observations, to investigate n-Lie algebras (any $n \geqslant 3$) with invariant symmetric (nondegenerate or not) bilinear forms.

This paper is organized as follows. In section 2, we recall some notations and facts on n-Lie algebras. In particular, we summarize the classification of real simple n-Lie algebras. In section 3 we give the concrete bracket structures of all real simple n-Lie algebras of type I and the real simple 3-Lie algebra of type II, and prove that the metric dimension of a real simple n-Lie algebra is equal to 1 or 2 depending on that it is of type I or II. In section 4 we construct an example of real 3-Lie algebras which is indecomposable but has much larger metric dimension.

Throughout this paper, K will denote the number field $\mathbb{R}$ or $\mathbb{C}$ and all n-Lie algebras will be finite dimensional over K.

2. Some results on *n*-Lie algebras

We recall in this section some notations and results on n-Lie algebras which can be found in [11] and [12].

Let A be a vector space over K. An n-Lie algebra structure on A consists of a linear map $[, \ldots,]\colon \wedge^n A \to A$ such that the generalized Jacobi identity

$$[x_1, \ldots, x_{n-1}, [y_1, \ldots, y_n]] = \sum_{i=1}^{n} [y_1, \ldots, [x_1, \ldots, x_{n-1}, y_i], \ldots, y_n]$$

holds, for all $x_k, y_k \in A$.

A subspace I of an n-Lie algebra A is called an ideal of A if $[I, A, \ldots, A] \subseteq I$. Clearly, the center $Z(A) = \{x \in A \mid [x, A, \ldots, A] = 0\}$ is an ideal of A. Call A solvable if $A^{(k)} = 0$ for some k, where $A^{(1)} = [A, \ldots, A]$, and $A^{(l+1)} = [A^{(l)}, \ldots, A^{(l)}]$. Call A nilpotent if $A^k = 0$ for some k, where $A^1 = [A, \ldots, A]$, and $A^{l+1} = [A^l, A, \ldots, A]$. The unique maximal solvable ideal of A is called the radical of A and denoted by $\mathrm{Rad}A$. If $\mathrm{Rad}A = 0$, A is called semisimple. If A has no ideals except itself and 0, and if moreover $A^1 = [A, \ldots, A] \neq 0$, A is called simple. A is called Abelian, if $A^1 = [A, \ldots, A] = 0$, A is called perfect if $A^1 = A$.

Proposition 2.1 ([12])**.** *There is, up to isomorphism, a unique simple n-Lie algebra over* $\mathbb{C}$*, for every* $n > 2$*. This algebra is of dimension* $(n + 1)$ *and its bracket is given relative to a basis* $\{e_1, \ldots, e_{n+1}\}$ *by*

$$[e_1, \ldots, \hat{e}_i, \ldots, e_{n+1}] = (-1)^{n+1+i} e_i, \qquad i = 1, 2, \ldots, n + 1, \tag{2.1}$$

where the symbol $\hat{e}_i$ *means that* e_i *is omitted.*

Remark 2.1. In the following, we denote by A_0 the unique complex simple n-Lie algebra given by (2.1).

Remark 2.2. In the following, any brackets not listed in a multiplication table of an n-Lie algebra are assumed to be equal to zero.

Proposition 2.2 ([11, 12]). *An n-Lie algebra $\tilde{A}$ over $\mathbb{C}$ or $\mathbb{R}$ is semisimple if and only if $\tilde{A}$ is a direct sum of simple ideals.*

In order to classify the finite-dimensional real simple n-Lie algebras, we will recall the definitions of complexification of a real n-Lie algebra, and of the realification of a complex n-Lie algebra.

Let $\tilde{A}$ be an arbitrary real n-Lie algebra. We set up the tensor product $\tilde{A}^{\mathbb{C}} := \tilde{A} \otimes \mathbb{C}$ and regard it as a vector space over $\mathbb{C}$ by $z_1(x \otimes z_2) := x \otimes z_1 z_2$, for $x \in \tilde{A}$, and $z_1, z_2 \in \mathbb{C}$. Obviously, $\tilde{A}^{\mathbb{C}}$ is an n-Lie algebra with bracket

$$[x_1 \otimes z_1, \ldots, x_n \otimes z_n] := [x_1, \ldots, x_n] \otimes z_1 \cdots z_n.$$

This complex n-Lie algebra is called the complexification of $\tilde{A}$. Clearly, $\dim_{\mathbb{R}} \tilde{A} = \dim_{\mathbb{C}} \tilde{A}^{\mathbb{C}}$.

Conversely, given a complex n-Lie algebra $\tilde{A}$, then by restricting the ground field $\mathbb{C}$ to $\mathbb{R}$, we obtain a real n-Lie algebra, which will be called the realification of $\tilde{A}$ and denoted by $\tilde{A}_{\mathbb{R}}$. Clearly, if $\{e_1, \ldots, e_m\}$ is a basis of $\tilde{A}$, then $\{e_1, \ldots, e_m, \sqrt{-1}e_1, \ldots, \sqrt{-1}e_m\}$ is a basis of $\tilde{A}_{\mathbb{R}}$, so $\dim_{\mathbb{R}} \tilde{A}_{\mathbb{R}} = 2\dim_{\mathbb{C}} \tilde{A}$.

A real n-Lie algebra $\tilde{A}$ is called a real form of a complex n-Lie algebra $\tilde{A}$ if $\tilde{A}^{\mathbb{C}}$ is isomorphic to $\tilde{A}$.

Proposition 2.3 ([11]). *Let $\tilde{A}$ be an arbitrary complex simple n-Lie algebra. Then the realification $\tilde{A}$ of $\tilde{A}_{\mathbb{R}}$ is simple.*

Proposition 2.4 ([11]). *A real simple n-Lie algebra is either isomorphic to the realification of a complex simple n-Lie algebra, or isomorphic to a real form of a complex simple n-Lie algebra.*

According to proposition 2.4, in order to find all real simple n-Lie algebras ($n \geqslant 3$), it suffices to calculate the real forms and the realifications of all complex simple n-Lie algebras. However, as proposition 2.1 shows, there is up to isomorphism only one finite-dimensional complex simple n-Lie algebra A_0 of dimension (n+1) given by remark 2.1. Therefore, by proposition 2.3, as one simple real n-Lie algebra we have the realification of A_0 and it is of dimension 2(n+1) over $\mathbb{R}$. On the other hand, it has been shown that (see [11, 12]) A_0 has up to isomorphism $\left[\frac{n+1}{2}\right] + 1$ real forms and every real form is determined by a bracket table given by (3.2) below. In summary, we have arrived at the following classification theorem of real simple n-Lie algebras.

Theorem 2.5. *All real simple n-Lie algebras are divided into two types:*

type I: $\left[\frac{n+1}{2}\right] + 1$ n-Lie algebras of dimension $n + 1$,
type II: one n-Lie algebra of dimension $2(n + 1)$.

3. Real simple *n*-Lie algebras

Let A be an arbitrary n-Lie algebra over K. A bilinear form b on A is said to be invariant if $b([x_1, \ldots, x_{n-1}, x], y) = -b(x, [x_1, \ldots, x_{n-1}, y])$ for all $x_i, x, y \in A$. A nondegenerate symmetric invariant bilinear form b on A is called a metric (or an inner product) and (A, b) is said to be a metric n-Lie algebra.

Let us denote by $F(A)$ the vector space of all symmetric invariant bilinear forms on A and let $B(A)$ be the subspace of $F(A)$ spanned by all metrics. We will call $\dim B(A)$ the metric dimension of A and we will say that A admits a unique (up to a constant) metric structure if $\dim B(A) = 1$.

Lemma 3.1 (cf [13]). *If A is an n-Lie algebra over K admitting a metric, then $B(A) = F(A)$.*

Proof. Let b a metric on A and $\varphi \in F(A)$. Denote by $M(b)$ and $M(\varphi)$ the matrices associated with b and φ relative to a basis of A, respectively. Observe that the determinant $\det(M(\varphi) - \lambda M(b))$ is a polynomial in λ. We may find $\lambda_0 \in K, \lambda_0 \neq 0$, such that $\det(M(\varphi) - \lambda M(b)) \neq 0$. This shows that $\varphi - \lambda_0 b$ is nondegenerate and $\varphi = (\varphi - \lambda_0 b) + \lambda_0 b \in B(A)$. □

Lemma 3.2. *Let A be an n-Lie algebra over K and let $A = I_1 \oplus \cdots \oplus I_m$, where each I_k is an ideal of A.*

(i) $\dim F(A) \geqslant \sum_{k=1}^{m} \dim F(I_k)$.
(ii) If, in addition, there are at least $(m-1)$ ideals among $I_1, \ldots, I_m$ which are perfect, then $\dim F(A) = \sum_{k=1}^{m} \dim F(I_k)$.

Proof. One may consider $F(I_k)$ as a subspace of $F(A)$ by extending any $b_k \in F(I_k)$ by zero in a natural way. Hence, $F(A) \supseteq \bigoplus_{k=1}^{m} F(I_k)$ which proves (i). In order to prove (ii), we assume that I_l is perfect. Then for any $\varphi \in F(A)$ and $l' \neq l$, we have

$$\varphi(I_l, I_{l'}) = \varphi([I_l, \ldots, I_l], I_{l'}) = \varphi(I_l, [I_l, \ldots, I_l, I_{l'}]) = 0.$$

Moreover, since $\varphi\mid_{I_k \times I_k} \in F(I_k)$ for any k, $1 \leqslant k \leqslant m$, it follows that $\varphi = \sum_{k=1}^{m} \varphi\mid_{I_k \times I_k}$, and hence $F(A) = \bigoplus_{k=1}^{m} F(I_k)$, which completes the proof. □

Lemma 3.3. *Let A be an n-Lie algebra over $\mathbb{R}$, then*

$$\dim_{\mathbb{R}} F(A) \leqslant \dim_{\mathbb{C}} F(A^{\mathbb{C}}).$$

Proof. Suppose that $\{\varphi_i \mid 1 \leqslant i \leqslant l\}$ are linearly independent elements in $F(A)$. Extend each form φ_i to $A^{\mathbb{C}}$ by

$$\bar{\varphi}_i(x_1 \otimes z_1, x_2 \otimes z_2) = z_1 z_2 \varphi_i(x_1, x_2), \qquad x_i \in A, \qquad z_i \in \mathbb{C}.$$

It is easy to see that each $\bar{\varphi}_i \in F(A^{\mathbb{C}})$. We now prove that $\{\bar{\varphi}_i \mid 1 \leqslant i \leqslant l\}$ are linearly independent over $\mathbb{C}$. Assume that there exist $\lambda_i \in \mathbb{C}, 1 \leqslant i \leqslant l$, such that $\sum_{i=1}^{l} \lambda_i \bar{\varphi}_i(x_1 \otimes z_1, x_2 \otimes z_2) = 0$ for any $x_i \in A$ and $z_i \in \mathbb{C}$. In particular, $\sum_{i=1}^{l} \lambda_i \bar{\varphi}_i(x_1 \otimes 1, x_2 \otimes 1) = \sum_{i=1}^{l} \lambda_i \varphi_i(x_1, x_2) = 0$. Suppose that $\lambda_i = a_i + \sqrt{-1} b_i, a_i, b_i \in \mathbb{R}$, then $\sum_{i=1}^{l} a_i \varphi_i(x_1, x_2) = \sum_{i=1}^{l} b_i \varphi_i(x_1, x_2) = 0$. Hence, $a_i = b_i = 0$, i.e. $\lambda_i = 0$, which implies that $\dim_{\mathbb{R}} F(A) \leqslant \dim_{\mathbb{C}} F(A^{\mathbb{C}})$. □

We now investigate the metric structures of all real simple n-Lie algebras ($n \geqslant 3$).

First we consider the real simple n-Lie algebras of type I. Let $\{e_1, \ldots, e_{n+1}\}$ be a basis of the vector space $\mathbb{R}^{n+1}$ and b_s, $0 \leqslant s \leqslant \left[\frac{n+1}{2}\right]$, be a nondegenerate symmetric bilinear form on $\mathbb{R}^{n+1}$ whose associated matrix relative to this basis is $\mathrm{diag}(-1, \ldots, -1, 1, \ldots, 1)$ with s negatives, and f be a nonzero determinant form satisfying $f(e_1, \ldots, e_{n+1}) = 1$. For $v_1, \ldots, v_n \in \mathbb{R}^{n+1}$ let $[v_1, \ldots, v_n]_s$ be the unique element in $\mathbb{R}^{n+1}$ such that for all $x \in \mathbb{R}^{n+1}$ the identity

$$b_s([v_1, \ldots, v_n]_s, x) = f(v_1, \ldots, v_n, x) \tag{3.1}$$

holds. Then with the bracket $[v_1, \ldots, v_n]_s$, $\mathbb{R}^{n+1}$ becomes an n-Lie algebra, which will be denoted by A_{n+1}^s. According to [11], $A_{n+1}^s \cong A_{n+1}^{s'}$ if and only if $s = s'$, $0 \leqslant s \leqslant \left[\frac{n+1}{2}\right]$, and $\left\{A_{n+1}^s \middle| 0 \leqslant s; s' \leqslant \left[\frac{n+1}{2}\right]\right\}$ exhaust all real simple n-Lie algebras of type I.

Let $[e_1, \ldots, \hat{e}_i, \ldots, e_{n+1}]_s = \sum_{k=1}^{n+1} a_k e_k, a_k \in \mathbb{R}$, then

$$b_s([e_1, \ldots, \hat{e}_i, \ldots, e_{n+1}]_s, e_j) = \begin{cases} -a_j, & j \leqslant s, \\ a_j, & j > s. \end{cases}$$

Note that

$$f(e_1, \ldots, \hat{e}_i, \ldots, e_{n+1}, e_j) = \begin{cases} 0, & i \neq j, \\ (-1)^{n+1-j}, & i = j. \end{cases}$$

Then by (3.1), we have

$$[e_1, \ldots, \hat{e}_i, \ldots, e_{n+1}]_s = \begin{cases} (-1)^{n-i} e_i, & i \leqslant s, \\ (-1)^{n+1-i} e_i, & i > s. \end{cases} \tag{3.2}$$

Identity (3.2) is the bracket table of real simple n-Lie algebra A_{n+1}^s.

Theorem 3.4. *Let A_{n+1}^s, b_s as above. Then*

(i) $\varphi \in F(A_{n+1}^s)$ *if and only if* $\varphi = \alpha b_s$ *for some* $\alpha \in \mathbb{R}$,
(ii) $\dim F(A_{n+1}^s) = \dim B(A_{n+1}^s) = 1$.

Proof. Since f is a determinant form, it follows that every symmetric bilinear form b_s is invariant. Indeed, according to (3.1), we have for any $v_1, \ldots, v_n, x \in \mathbb{R}^{n+1}$,

$$\begin{aligned} b_s([v_1, \ldots, v_n], x) &= f(v_1, \ldots, v_n, x) = -f(v_1, \ldots, v_{n-1}, x, v_n) \\ &= -b_s([v_1, \ldots, v_{n-1}, x], v_n) = -b_s(v_n, [v_1, \ldots, v_{n-1}, x]). \end{aligned}$$

Thus, $\alpha b_s \in F(A_{n+1}^s)$ for any $\alpha \in \mathbb{R}$. Conversely, suppose $\varphi \in F(A_{n+1}^s)$. Using (3.2) one has the following:

$$\begin{aligned} \text{if} \quad i \neq j, \qquad \varphi(e_i, e_j) &= \pm\varphi([e_1, \ldots, \hat{e}_i, \ldots, e_{n+1}]_s, e_j) \\ &= \pm\varphi(e_{n+1}, [e_1, \ldots, e_j, \ldots, e_j]_s) = 0, \\ \text{if} \quad i \leqslant s, \qquad \varphi(e_i, e_i) &= (-1)^{n-i}\varphi([e_1, \ldots, \hat{e}_i, \ldots, e_{n+1}]_s, e_i) \\ &= (-1)^{n-i+1}\varphi(e_{n+1}, [e_1, \ldots, \hat{e}_i, \ldots, e_n, e_i]_s) \\ &= -\varphi(e_{n+1}, [e_1, \ldots, e_n]_s) = -\varphi(e_{n+1}, e_{n+1}), \\ \text{if} \quad i > s, \qquad \varphi(e_i, e_i) &= (-1)^{n+1-i}\varphi([e_1, \ldots, \hat{e}_i, \ldots, e_{n+1}]_s, e_i) \\ &= (-1)^{n-i}\varphi(e_{n+1}, [e_1, \ldots, \hat{e}_i, \ldots, e_n, e_i]_s) \\ &= \varphi(e_{n+1}, [e_1, \ldots, e_n]_s) = \varphi(e_{n+1}, e_{n+1}). \end{aligned}$$

Thus, we have

$$\varphi(e_i, e_j) = \begin{cases} 0, & i \neq j, \\ -\varphi(e_{n+1}, e_{n+1}), & i = j \leqslant s, \\ \varphi(e_{n+1}, e_{n+1}), & i = j > s. \end{cases}$$

Since

$$b_s(e_i, e_j) = \begin{cases} 0, & i \neq j, \\ -1, & i = j \leqslant s, \\ 1, & i = j > s. \end{cases}$$

It follows that $\varphi = \alpha b_s$ for some $\alpha \in \mathbb{R}$, which proves assertion (i). Assertion (ii) follows at once from lemma 3.1 and assertion (i). □

Remark 3.1. Here our 3-Lie algebra A_4^0 is denoted by A_4 in some articles (e.g. [6]).

Next we study the real simple n-Lie algebras of type II. For simplicity, we only consider the case $n = 3$.

Let A_0 be the unique complex simple 3-Lie algebra with bracket table

$$[e_1, \ldots, \hat{e_i}, \ldots, e_4] = (-1)^i e_i, \qquad i = 1, 2, 3, 4. \tag{3.3}$$

Then the realification, denoted by A_8, of A_0 is a real simple 3-Lie algebra of dimension 8, and $\{e_i, e_{4+i} := \sqrt{-1}e_i \mid 1 \leqslant i \leqslant 4\}$ is a basis of A_8. From (3.3), we can write out the bracket table of A_8 as follows:

$$\begin{aligned} &[e_1, \ldots, \hat{e_i}, \ldots, e_4] = (-1)^i e_i, \\ &[e_{4+1}, \ldots, \hat{e_{4+i}}, \ldots, e_{4+4}] = (-1)^{i+1} e_{4+i}, \qquad 1 \leqslant i \leqslant 4, \\ &[e_i, e_j, e_{4+k}] = \sqrt{-1}[e_i, e_j, e_k], \qquad 1 \leqslant i < j, \qquad k \leqslant 4, \\ &[e_i, e_{4+j}, e_{4+k}] = -[e_i, e_j, e_k], \qquad 1 \leqslant i, j < k \leqslant 4. \end{aligned} \tag{3.4}$$

Note that A_8 is the unique real simple 3-Lie algebra of type II. Using identity (3.4), one can easily verify that if φ is an invariant symmetric bilinear form on A_8 then φ satisfies that

$$\varphi(e_i, e_j) = 0, 1 \leqslant i, \qquad j \leqslant 8,$$

except for

$$\varphi(e_1, e_1) = \cdots = \varphi(e_4, e_4) = -\varphi(e_5, e_5) = \cdots = -\varphi(e_8, e_8)$$

and

$$\varphi(e_1, e_5) = \varphi(e_2, e_6) = \varphi(e_3, e_7) = \varphi(e_4, e_8).$$

Define two bilinear forms φ_1 and φ_2 on A_8 given by

$$\varphi_1(e_i, e_j) = 0, 1 \leqslant i \neq j \leqslant 8, \qquad \varphi_1(e_i, e_i) = -\varphi(e_{4+i}, e_{4+i}) = 1, \qquad 1 \leqslant i \leqslant 4,$$
$$\varphi_2(e_i, e_j) = 0, 1 \leqslant i, j \leqslant 8, \qquad \text{except for} \quad \varphi_2(e_i, e_{4+i}) = 1, \qquad 1 \leqslant i \leqslant 4.$$

Clearly, φ_1 and φ_2 are linearly independent metrics on A_8.

Theorem 3.5.

(i) $\dim F(A_8) = \dim B(A_8) = 2$.
(ii) Any metric on A_8 has the signature $(4, 4)$.

Proof.

(i) By proposition 2.4, $A_8^{\mathbb{C}}$, the complexification of A_8, is semisimple, so $A_8^{\mathbb{C}} = A_0 \oplus A_0$, where A_0 is the unique complex simple 3-Lie algebra of dimension 4. Using lemmas 3.2 and 3.3, one has $\dim_{\mathbb{R}} F(A_8) \leqslant \dim_{\mathbb{C}} F(A_8^{\mathbb{C}}) = 2\dim_{\mathbb{C}} F(A_0)$. Since A_0 is simple, similar argument shows that $\dim_{\mathbb{C}} F(A_0) = \dim_{\mathbb{C}} B(A_0) = 1$. On the other hand, since φ_1 and φ_2 are linearly independent, $\dim_{\mathbb{R}} F(A_8) \geqslant 2$. Therefore, $\dim F(A_8) = \dim B(A_8) = 2$, by lemma 3.1.

(ii) Let $\varphi = k_1\varphi_1 + k_2\varphi_2$ be an arbitrary metric on A_8, where φ_i are given as above and $k_i \in \mathbb{R}, i = 1, 2$. According to the knowledge of symmetric forms in linear algebra, we can prove by a direct calculation that the associated matrix of φ relative to $\{e_1, \ldots, e_8\}$ is congruent to $\mathrm{diag}(1, 1, 1, 1, -1, -1, -1, -1)$. In other words, relative to another basis of A_8, the associated matrix of φ is $\mathrm{diag}(1, 1, 1, 1, -1, -1, -1, -1)$. □

Remark 3.2. Theorem 3.5 is true for any $n \geqslant 3$.

To conclude this section, we observe the metric structure on a real reductive n-Lie algebra. Assume that A is a real reductive n-Lie algebra, i.e. $A = S \oplus Z(A)$, where S is a semisimple ideal and $Z(A)$ is the center of A. Suppose that S is a direct sum of l_1 simple ideals of type I and l_2 simple ideals of type II, and $\dim Z(A) = r$.

Theorem 3.6. *Let A be a real reductive n-Lie algebra as above. Then A admits a metric and* $\dim B(A) = l_1 + 2l_2 + \frac{r(r+1)}{2}$.

Proof. We first find a metric on A. Let $Z(A) = \langle e_1, \dots, e_r\rangle$. Define a bilinear form φ on $Z(A)$ given by $\varphi(e_i, e_j) = \delta_{ij}, 1 \leqslant i, j \leqslant r$. φ is clearly a metric on $Z(A)$. Note that S is a direct sum of $l_1 + l_2$ simple ideals and each of them admits a metric, say φ_i. Then we may construct a metric on A from $\varphi_1, \dots, \varphi_{l_1+l_2}$ and φ in a natural way. Hence, $F(A) = B(A)$ by lemma 3.1. By lemma 3.2, one has $\dim F(A) = \dim F(S) + \dim F(Z(A))$. Moreover by theorems 3.4 and 3.5, $\dim F(S) = \dim B(S) = l_1 + 2l_2$. On the other hand, $\dim F(Z(A)) = \frac{r(r+1)}{2}$ since the $\frac{r(r+1)}{2}$ symmetric bilinear forms φ_{ij} on $Z(A)$ defined to be zero except for $\varphi_{ij}(e_i, e_j) = 1, 1 \leqslant i, j \leqslant r$, are invariant and form a basis of $F(Z(A))$. Therefore, $\dim F(A) = \dim B(A) = l_1 + 2l_2 + \frac{r(r+1)}{2}$. □

4. An example of 3-Lie algebras

We have proved that any real simple n-Lie algebra has metric dimension 1 or 2. Now we present an example of real 3-Lie algebras which is indecomposable but has much larger metric dimension. This example is inspired by a result in [13].

We begin with recalling some facts on 3-Lie algebras. Let A be an arbitrary 3-Lie algebra over $\mathbb{C}$ or $\mathbb{R}$. If $\dim A < 3$, then A is Abelian. If $\dim A = 3$, then either A is Abelian, or A has a basis $\{e_1, e_2, e_3\}$ such that $[e_1, e_2, e_3] = e_1$ which shows that A is not nilpotent. Moreover, according to the multiplication table of n-Lie algebras of dimension $n+1$ (see [12]), there is a unique nilpotent and non-Abelian 3-Lie algebra of dimension 4 whose bracket is given, relative to a basis $\{e_1, \dots, e_4\}$, by $[e_2, e_3, e_4] = e_1$, which shows that A is nilpotent and non-Abelian. Summarizing above argument we obtain the following:

Lemma 4.1 (cf [12])**.** *Let A be a nilpotent 3-Lie algebra over* $\mathbb{C}$ *or* $\mathbb{R}$*. If* $\dim A \leqslant 3$*, A is Abelian. If* $\dim A = 4$*, A is non-Abelian and the center* $Z(A)$ *is of dimension 1.*

From now on, let A denote a real simple 3-Lie algebra of type I, i.e. $A = A_4^s$, $s = 0, 1, 2$. *Consider the vector space* $N(A, 2) := AT_1 \oplus AT_2$, *where* $AT_i, i = 1, 2$, *is a copy of A. Define the bracket on* $N(A, 2)$ *by*

$$[xT_i, yT_j, zT_k] = \begin{cases} [x, y, z]T_{i+j+k-1}, & i+j+k-1 \leqslant 2,\\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $N(A, 2)$ *is a real 3-Lie algebra of dimension 8.*

Lemma 4.2. *Let* $N = N(A, 2)$ *be as above.*

(i) N is nilpotent. More precisely, $N^1 = Z(N) = AT_2$, $N^2 = 0$.
(ii) N admits a metric.

Proof. Assertion (i) is obvious. In order to prove (ii), we assume that b is a metric on A, and define a bilinear form on N by $\bar{b}(xT_i, yT_j) = b(x, y)\delta_3^{i+j}, x, y \in A$, where δ_s^t is the Kronecker symbol. Clearly, $\bar{b}$ is symmetric since b is so. To prove that $\bar{b}$ is nondegenerate, we take a nonzero vector $X = x_1T_1 + x_2T_2 \in N$. Without loss of generality we may assume that $x_1 \neq 0$. Since b is nondegenerate, there exists $y \in A$ such that $b(x_1, y) \neq 0$. Hence, $\bar{b}(X, yT_2) = b(x_1, y) \neq 0$. Now we prove that $\bar{b}$ is invariant. Let $x, y, u, v \in A$, then $\bar{b}([xT_i, yT_j, uT_k], vT_l) = \bar{b}([x, y, u]T_{i+j+k-1}, vT_l) = b([x, y, u], v)\delta_3^{i+j+k+l-1} = -b\big(u, [x, y, v]\delta_3^{i+j+k+l-1}\big) = -\bar{b}(uT_k, [xT_i, yT_j, vT_l]), i, j, k, l = 1, 2$, which completes the proof. □

Theorem 4.3. *$N = N(A, 2)$ as above.*

(i) N is indecomposable.
(ii) $\dim B(N) \geqslant 5$.

Proof.

(i) Suppose that $N = I \oplus J$, where I and J are non-zero ideals of N. If one of them, say I, has dimension greater than 4, then $\dim J \leqslant 3$.

Lemma 4.2 shows that J is nilpotent, so J must be Abelian by lemma 4.1. Thus, one has that $[J, N, N] \subseteq [J, J, J] = 0$, which shows that $J \subseteq Z(N)$, the center of N. Thus, by lemma 4.2, one has that

$$J \subseteq Z(N) = N^1 = [N, N, N] = [I, I, I] \subseteq I,$$

which is impossible since $I \cap J = \{0\}$. On the other hand, if $\dim I = \dim J = 4$, then again by lemma 4.1, I and J are all non-Abelian and $\dim Z(I) = \dim Z(J) = 1$. But by lemma 4.2, $\dim Z(N) = 4$, which leads to a contradiction for $Z(N) = N(I) \oplus N(J)$. This shows that N must be indecomposable.

(ii) In order to prove that $\dim B(N) \geqslant 5$, it suffices to prove that $\dim F(N) \geqslant 5$ according to lemmas 4.2 and 3.1. Let $\{e_1, \dots, e_4\}$ be a basis of A, then $\{x_i = e_iT_1, x_{4+i} = e_iT_2 \mid 1 \leqslant i \leqslant 4\}$ is a basis of N such that $\{x_5, \dots, x_8\}$ provides a basis of $Z(N)$. Let $\bar{b}$ be the metric on N given by lemma 4.2. For each $j = 1, 2, 3, 4$ define a bilinear form φ_j on N by

$$\varphi_j\left(\sum_{i=1}^{8} a_i x_i \sum_{i=1}^{8} b_i x_i\right) = a_j b_j, \qquad a_i, b_i \in \mathbb{R}.$$

It is easy to prove that φ_j is invariant and symmetric. We now prove that the bilinear forms $\bar{b}$ and φ_j, $j = 1, 2, 3, 4$, are linearly independent. Take $k_j \in \mathbb{R}$, $j = 0, 1, 2, 3, 4$, such that $k_0\bar{b} + \sum_{j=1}^{4} k_j\varphi_j = 0$. Since b is nondegenerate on A, there are $u, v \in A$, such that $b(u, v) \neq 0$. Thus,

$$\left(k_0\bar{b} + \sum_{j=1}^{4} k_j\varphi_j\right)(uT_1, vT_2) = k_0\bar{b}(uT_1, vT_2) = k_0 b(u, v),$$

which forces that $k_0 = 0$. Similarly, for $i = 1, 2, 3, 4$,

$$\left(k_0\bar{b} + \sum_{j=1}^{4} k_j\varphi_j\right)(x_i, x_i) = k_i\varphi_i(x_i, x_i) = k_i = 0.$$

Hence, $\bar{b}$ and φ_j, $1 \leqslant j \leqslant 4$, are linearly independent and $\dim F(N) \geqslant 5$. □

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